

A characterization of geodesic hyperspheres of quaternionic projective space

著者	Perez Juan de Dios
journal or publication title	Tsukuba journal of mathematics
volume	21
number	1
page range	113-116
year	1997-06
URL	http://hdl.handle.net/2241/100420

A CHARACTERIZATION OF GEODESIC HYPERSPHERES OF QUATERNIONIC PROJECTIVE SPACE

By

Juan de Dios PÉREZ

Abstract. We study a condition that allows us to characterize geodesic hyperspheres among all real hypersurfaces of quaternionic projective space.

1. Introduction

Along this paper M will denote a connected real hypersurface of the quaternionic projective space QP^m , $m \geq 3$, endowed with the metric g of constant quaternionic sectional curvature 4. Let N be a unit local normal vector field on M and $U_i = -J_i N$, $i = 1, 2, 3$, where $\{J_i\}_{i=1,2,3}$ is a local basis of the quaternionic structure of QP^m , [2]. Let us denote by $D^\perp = \text{Span}\{U_1, U_2, U_3\}$ and by D its orthogonal complement in TM .

If A denotes the Weingarten endomorphism of M we have the

THEOREM A, [1]. *Let M be a real hypersurface of QP^m , $m \geq 2$. Then $g(AD, D^\perp) = \{0\}$ if and only if M is congruent to an open part of one of the following real hypersurfaces of QP^m :*

- i) *a geodesic hypersphere,*
- ii) *a tube of some radius r , $0 < r < \pi/2$, around the canonically (totally geodesic) embedded quaternionic projective space QP^k , $k \in \{1, \dots, m-2\}$,*
- iii) *a tube of some radius r , $0 < r < \pi/4$, around the canonically (totally geodesic) embedded projective space CP^m .*

Let us denote by R the curvature tensor of M . In [4] we have proved that there do not exist real hypersurfaces of QP^m , $m \geq 2$, such that $\sigma(R(X, Y)AZ) = 0$, for any X, Y, Z tangent to M , where σ denotes the cyclic sum.

The purpose of the present paper is to study a weaker condition than the one considered in [4]. Concretely we propose to study real hypersurfaces of QP^m such that

$$(1.1) \quad \sigma(R(X, Y)AZ) = 0$$

for any $X, Y, Z \in \mathcal{D}$. We shall prove the following

THEOREM. *Let M be a real hypersurface of QP^m , $m \geq 3$. Then M satisfies (1.1) if and only if it is congruent to an open part of a geodesic hypersphere of QP^m .*

2. Preliminaries

Let X be a tangent vector field to M . We write $J_i X = \phi_i X + f_i(X)N$, $i = 1, 2, 3$, where $\phi_i X$ is the tangent component of $J_i X$ and $f_i(X) = g(X, U_i)$, $i = 1, 2, 3$. As $J_i^2 = -Id$, $i = 1, 2, 3$, where Id denotes the identity endomorphism on TQP^m , we get

$$(2.1) \quad \phi_i^2 X = -X + f_i(X)U_i, \quad f_i(\phi_i X) = 0, \quad \phi_i U_i = 0, \quad i = 1, 2, 3$$

for any X tangent to M . As $J_i J_j = -J_j J_i = J_k$, where (i, j, k) is a cyclic permutation of $(1, 2, 3)$ we obtain

$$(2.2) \quad \phi_i X = \phi_j \phi_k X - f_k(X)U_j = -\phi_k \phi_j X + f_j(X)U_k$$

and

$$(2.3) \quad f_i(X) = f_j(\phi_k X) = -f_k(\phi_j X)$$

for any X tangent to M , where (i, j, k) is a cyclic permutation of $(1, 2, 3)$. It is also easy to see that for any X, Y tangent to M and $i = 1, 2, 3$,

$$(2.4) \quad g(\phi_i X, Y) + g(X, \phi_i Y) = 0, \quad g(\phi_i X, \phi_i Y) = g(X, Y) - f_i(X)f_i(Y)$$

and

$$(2.5) \quad \phi_i U_j = -\phi_j U_i = U_k$$

(i, j, k) being a cyclic permutation of $(1, 2, 3)$. Finally from the expression of the curvature tensor of QP^m , $m \geq 2$, we have that the curvature tensor of M is given

by

$$(2.6) \quad R(X, Y)Z = g(Y, Z)X - g(X, Z)Y + \sum_{i=1}^3 \{g(\phi_i Y, Z)\phi_i X - g(\phi_i X, Z)\phi_i Y \\ + 2g(X, \phi_i Y)\phi_i Z\} + g(AY, Z)AX - g(AX, Z)AY$$

for any X, Y, Z tangent to M , see [3].

3. Proof of the Theorem

Let $\{E_1, \dots, E_{4m-4}\}$ be an orthonormal basis of D at any point of M .

If in (1.1) we take $Z = E_j$, $Y = \phi_1 E_j$, from (2.6) and applying the formulas (2.1) to (2.5) we have for any $X \in D$

$$(3.1) \quad \{g(\phi_1 X, AE_j) - g(AX, \phi_1 E_j)\}E_j + \{g(AX, E_j) + g(\phi_1 X, A\phi_1 E_j)\}\phi_1 E_j \\ + \{2g(AX, \phi_3 E_j) - g(\phi_3 X, AE_j) + g(\phi_2 X, A\phi_1 E_j)\}\phi_2 E_j + \{g(\phi_2 X, AE_j) \\ + g(\phi_3 X, A\phi_1 E_j) - 2g(AX, \phi_2 E_j)\}\phi_3 E_j - 2g(X, E_j)\phi_1 AE_j \\ - 2g(X, \phi_3 E_j)\phi_2 AE_j + 2g(X, \phi_2 E_j)\phi_3 AE_j + 2g(\phi_1 X, E_j)\phi_1 A\phi_1 E_j \\ + 2g(\phi_2 X, E_j)\phi_2 A\phi_1 E_j + 2g(\phi_3 X, E_j)\phi_3 A\phi_1 E_j - \{g(E_j, AE_j) \\ + g(\phi_1 E_j, A\phi_1 E_j)\}\phi_1 X - \{g(\phi_3 E_j, AE_j) + g(\phi_2 E_j, A\phi_1 E_j)\}\phi_2 X \\ + \{g(\phi_2 E_j, AE_j) - g(\phi_3 E_j, A\phi_1 E_j)\}\phi_3 X + 2\phi_1 AX = 0$$

If now we take the scalar product of (3.1) and U_1 and sum on j we obtain

$$(3.2) \quad g(\phi_2 X, AU_2) + g(\phi_3 X, AU_3) = 0$$

for any $X \in D$.

The same reasoning taking in (1.1) $Z = E_j$, $Y = \phi_2 E_j$ and considering the scalar product of the result and U_2 gives us

$$(3.3) \quad g(\phi_1 X, AU_1) + g(\phi_3 X, AU_3) = 0$$

for any $X \in D$.

If we repeat the above computation for $Z = E_j$, $Y = \phi_3 E_j$ and take the U_3 -component we get

$$(3.4) \quad g(\phi_1 X, AU_1) + g(\phi_2 X, AU_2) = 0$$

for any $X \in D$. Thus from (3.2), (3.3) and (3.4) we have

$$(3.5) \quad g(\phi_i X, AU_i) = 0, \quad i = 1, 2, 3$$

for any $X \in D$. Thus $g(AD, D^\perp) = \{0\}$ and from Theorem A, M must be congruent to an open part of either i), ii) or iii) appearing in such a Theorem.

Let us consider the case iii) of a tube of radius r , $0 < r < \pi/4$, over CP^m . The principal curvatures on D are $\cot(r)$ and $-\tan(r)$ both with multiplicity $2(m-1)$. As $m \geq 3$ we can consider unit $X, W \in D$ such that $\text{Span}\{X, \phi_1 X, \phi_2 X, \phi_3 X\} \perp \text{Span}\{W, \phi_1 W, \phi_2 W, \phi_3 W\}$ and such that X and $\phi_1 X$ are principal with principal curvature $\cot(r)$ and $\phi_2 W$ is principal with principal curvature $-\tan(r)$. Thus if in (1.1) we take $Y = \phi_1 X$ and $Z = \phi_2 W$, by the first identity of Bianchi we should have $-(\tan(r) + \cot(r))R(X, \phi_1 X)\phi_2 W = 0$. But applying (2.6) this implies $(\tan(r) + \cot(r))\phi_3 W = 0$ which is impossible.

In the case ii) of Theorem A we also have two distinct principal curvatures on D and a reasoning similar to the above one proves that this case cannot occur.

On the other hand, geodesic hyperspheres have only one principal curvature on D , thus they satisfy (1.1) and this finishes the proof.

References

- [1] J. Berndt, "Real hypersurfaces in quaternionic space forms", J. reine angew. Math., **419** (1991), 9–26.
- [2] S. Ishihara, "Quaternion Kählerian manifolds", J. Diff. Geom., **9** (1974), 483–500.
- [3] A. Martínez and J. D. Pérez, "Real hypersurfaces in quaternionic projective space", Ann. Mat. Pura Appl., **145** (1986), 355–384.
- [4] J. D. Pérez, "Some conditions on real hypersurfaces of quaternionic projective space", Anal. Stiint. Univ. "Al.I.Cuza", **38** (1992), 103–110.

Departamento de Geometría y Topología
Facultad de Ciencias. Universidad de Granada
18071-Granada, SPAIN.